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Entropy Function in the Liouville Theory

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Abstract In this paper, by using entropy function we calculate entropy of the two dimensional black hole in the Liouville theory. In this approach we couple the gauge field to the Liouville action. We also consider the possible higher derivative corrections in the Liouville action and find the modified black hole entropy.

Keywords Black hole · Entropy function · Liouville · Central charge

1 Introduction

The entropy function formalism [1] which may be derived from Wald's formula [2] is the useful method to calculate the entropy of black holes [3–16]. In Ref. [6] found that the entropy function formalism of Sen is coincide with the Euclidean formalism at zero temperature limit.

We would like to focus on two dimensional black holes. Since the horizon of such black holes is a point, so horizon area vanishes and we can't use ordinary method to find entropy, which is proportional to the horizon area. Therefore one can use entropy function formalism to obtain the black hole entropy. Already we have used entropy function formalism to obtain the entropy of a two dimensional black hole. We have shown that the entropy is proportional to the value of the dilaton field at the horizon which is expected [3]. Although the possibility of describing $2D$ black holes by means of a CFT has been widely investigated [17–23], it is not completely clear if it is always possible to mimic the gravitational dynamics of the 2D black hole through a CFT. However, in two-dimensional dilaton gravities [24–26], it has been shown that the entropy is proportional to the value of the dilaton field at the horizon. To find black hole entropy by using entropy function formalism one first rewrite the Lagrangian density in terms of value of fields near horizon, and then taking the Legendre transform of

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the resulting function with respect to the electric field with multiplication of an overall factor 2π. The near horizon geometry of the extremal black hole is determined by extremizing the entropy function and the black hole entropy is given by the extremum value of the entropy function. This general method is an easier way to calculate the black hole entropy.

Entropy function analysis provides a good understanding of the attractor mechanism for spherically symmetric extremal black holes if:

- We consider a theory of gravity coupled to Abelian (p-form) gauge fields and neutral scalar fields.
- The Lagrangian density f is gauge and general coordinate invariant.
- Define an extremal black hole to be one whose near horizon geometry is $AdS_2 \times S^2$ (in $D = 4$).

In this approach, the theory need not be supersymmetric and f could contain higher derivative terms. For such black holes one can define an 'entropy function' F as follows:

$$F = 2\pi[q_i \epsilon_i - f], \tag{1}$$

where q_i denote electric charges, and ϵ_i are near horizon radial electric field. F is a function of the q_i and various parameters labeling the $SO(2, 1) \times SO(3)$ symmetric near horizon background (e.g. sizes of AdS_2 and S^2, vacuum expectation value of scalars, radial electric fields, radial magnetic fields). Then for a black hole with given electric charges q and magnetic charges p, all other near horizon parameters are obtained by extremizing F with respect to these parameters. And finally the entropy is given by the value of F at its extremum. In that case there are many researches for extremal black hole such as [7–12]. In Ref. [8] the entropy function for the extremal Kerr-(anti) de Sitter black holes are obtained. However the entropy function formalism may be used for non-extremal black holes [13–15].

One of the important theory in two dimensions is the Liouville theory [27, 28]. Liouville theory naturally arises in the formulation of the two dimensional quantum gravity and in the path integral quantization of string theory [29]. This two dimensional theory comes from the dimensional reduction of higher dimensional theories. This is a non trivial CFT [30, 31] which action in this theory could be written as,

$$S = \frac{1}{4\pi} \int d^2 z \left(\partial\varphi\bar{\partial}\varphi + \frac{1}{2\sqrt{2}} QR\varphi + 4\pi\mu e^{\sqrt{2}b\varphi} \right), \tag{2}$$

where μ is real parameter so called the Liouville cosmological constant and $Q = b + b^{-1}$ is the background charge parameter, also $\mu e^{\sqrt{2}b\varphi}$ is the Liouville barrier potential. In the conformal gauge, the linear dilaton term $QR\varphi$, where R is two dimensional Ricci scalar, has to be understood as keeping track of the coupling with the world sheet curvature. We want to calculate entropy of black hole by using entropy function in the Liouville theory. We know that the entropy must be proportional to central charge of theory [32, 33]. The central charge of this theory is $c = 1 + 6Q^2$, in other words c is called Liouville central charge. In here we want to couple gauge field $\mathcal{F}$ to the action (2) in two ways and then obtain entropy of black hole by using entropy function from Wald's formula [1, 2].

An outline of paper is as follows: In Sect. 2 we couple gauge field $\mathcal{F}$ to action (2) as free field and obtain entropy of charged black hole. In Sect. 3 we couple gauge field $\mathcal{F}$ to the barrier potential in action (2) and obtain entropy. In both sections we consider effect of higher derivative terms and find modified entropy due to corrections. Finally in Sect. 4 we give the conclusion and compare our result with previous works.

2 Free Gauge Field

In this section we would like to calculate the entropy function for Liouville theory. Therefore the action (2) with coupling to a free gauge field strength $\mathcal{F}_{\mu\nu}$ is given by,

$$S=\frac{1}{4\pi}\int d^2z\left(\partial\varphi\bar{\partial}\varphi+\frac{1}{2\sqrt{2}}QR\varphi+4\pi\mu e^{\sqrt{2}b\varphi}-\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right). \tag{3}$$

The near horizon solution of the extremal black hole with charge q can be generically written in form of,

$$\begin{aligned} ds^2&=v\left(-g(r)dt^2+\frac{1}{g(r)}dr^2\right),\\ e^{\sqrt{2}b\varphi}&=u,\\ \mathcal{F}_{rt}&=\epsilon, \end{aligned} \tag{4}$$

where u, v and ϵ are constants can be determined in terms of the charge q, the Liouville cosmological constant μ and background charge parameter Q. Now we will construct Sen's entropy function [1] from Wald's entropy formula [2]. Therefore first by using (4) we write the Lagrangian density in terms of u, v and ϵ,

$$f(u,v,\epsilon)=\frac{1}{4\pi}\left[-\frac{Q}{2bv}\ln u+4\pi\mu u+\frac{\epsilon^2}{2v^2}\right]. \tag{5}$$

Then the entropy function is defined as the Legendre transform of the Lagrangian density with respect to the gauge field ϵ, which defined in (1)

$$F(u,v,q)=2\pi[q\epsilon-f(u,v,\epsilon)]=4\pi^2q^2v^2+\frac{Q}{4bv}\ln u-2\pi\mu u, \tag{6}$$

where $q=\frac{\partial f}{\partial\epsilon}=\frac{\epsilon}{4\pi v^2}$, denote the electric charge carried by the black hole. The undetermined parameter u and v can be fixed by the equations of motion, which becomes the extremum equations as,

$$\begin{aligned} \left.\frac{\partial F}{\partial v}\right|_{u_e,v_e}&=8\pi^2q^2v_e-\frac{Q}{4bv_e^2}\ln u_e=0,\\ \left.\frac{\partial F}{\partial u}\right|_{u_e,v_e}&=\frac{Q}{4bv_eu_e}-2\pi\mu=0, \end{aligned} \tag{7}$$

where u_e and v_e are extremum of $F(u,v,q)$.
The entropy is given by the value of the entropy function at the extremum,

$$S_{BH}=F(u_e,v_e,q)=\frac{Q}{4bv_e}\left(-1+\frac{3}{2}\ln u_e\right), \tag{8}$$

which is obtained by using (7) in (6). Also by solving both relations in (7) simultaneously, we find value of u_e in terms of Lambert function,

$$u_e=\exp\left(\frac{1}{3}LambertW\left(\frac{3q^2Q^2}{16\pi b^2\mu^3}\right)\right), \tag{9}$$

similarly the value of v_e is obtained by (7). Note that the entropy is proportional to Q^2 as is expected.

Now let us consider the effect of higher derivative terms. Since Riemann and Ricci tensors in two dimensions can be expressed in terms of Ricci scalar, it is sufficient to consider the higher derivative terms of the form R^n. Hence change of the action (3) due to higher derivative terms is,

$$\Delta S = \frac{1}{4\pi} \int d^2 z \frac{1}{2\sqrt{2}} Q \sum a_n R^n \varphi. \tag{10}$$

In here one can obtain entropy function as,

$$F(u, v, q) = 4\pi^2 q^2 v^2 + \frac{Q}{4bv} \ln u - 2\pi \mu u - \frac{Q}{8b} \sum a_n \left(\frac{-2}{v}\right)^n \ln u, \tag{11}$$

then entropy (8) is modified as,

$$\begin{aligned} S_{mod} = {} & \frac{Q}{4bv_e}\left(-1 + \frac{3}{2} \ln u_e\right) \\ & - \frac{Q}{8b}\left[-\sum a_n \left(\frac{-2}{v_e}\right)^n + \sum \left(\frac{n}{2} + 1\right) a_n \left(\frac{-2}{v_e}\right)^n \ln u_e\right]. \end{aligned} \tag{12}$$

In the next section we couple gauge field $\mathcal{F}$ to the potential in action (2).

3 Gauge Field Coupled to the Potential

In this section we consider action (2) which gauge field $\mathcal{F}$ coupled to the Liouville barrier potential, therefore we have,

$$S = \frac{1}{4\pi} \int d^2 z \left(\partial\varphi\bar{\partial}\varphi + \frac{1}{2\sqrt{2}} Q R \varphi - \pi \mu \mathcal{F}^2 e^{\sqrt{2} b \varphi}\right). \tag{13}$$

The near horizon solutions which has $SO(2,1)$ symmetry can be written as (4). In this case the Lagrangian density function is evaluated as,

$$f(u, v, \epsilon) = \frac{1}{4\pi}\left[-\frac{Q}{2bv} \ln u + 4\pi \mu \frac{\epsilon^2}{v^2} u\right], \tag{14}$$

and the electric charge is given by,

$$q = \mu \frac{\epsilon}{v^2} u. \tag{15}$$

So, the entropy function is,

$$F(u, v, q) = \frac{\pi q^2 v^2}{\mu u} + \frac{Q}{4bv} \ln u. \tag{16}$$

Now we extremize (16) with respect to v and u,

$$\left.\frac{\partial F}{\partial v}\right|_{u_e,v_e} = \frac{2\pi q^2 v_e}{\mu u_e} - \frac{Q}{4bv_e^2}\ln u_e = 0,$$
$$\left.\frac{\partial F}{\partial u}\right|_{u_e,v_e} = \frac{Q}{4bv_e u_e} - \frac{\pi q^2 v_e^2}{\mu u_e^2} = 0. \tag{17}$$

By using (17) and relation $Q = b + b^{-1}$ we obtain the black hole entropy as,

$$S_{BH} = F(u_e, v_e, q) = \frac{3Q}{8v_e}(Q-b)\ln u_e. \tag{18}$$

Here we obtained black hole entropy proportional to Q^2. Now we consider higher derivative correction terms. We have to add (10) to (13), the entropy function can be changed by following form,

$$F(u, v, q) = \frac{\pi q^2 v^2}{\mu u} + \frac{Q}{4bv}\ln u - \frac{Q}{8b}\sum a_n \left(\frac{-2}{v}\right)^n \ln u. \tag{19}$$

In that case the black hole entropy (18) is modified as,

$$S_{BH} = \frac{3Q}{8v_e}(Q-b)\ln u_e - \frac{Q}{8b}\left(\sum\left(\frac{n}{2}+1\right)a_n\left(\frac{-2}{v_e}\right)^n\right)\ln u_e. \tag{20}$$

4 Conclusion and Discussion

In this letter we denoted S_{BH} as entropy of a two-dimensional extremal charged black hole, with near horizon geometry $AdS_2 \times S^2$. The radial electric field ϵ at the horizon is the variable conjugate to the electric charge of black hole q. We calculated entropy function by Legendre transform of Lagrangian density with respect to the parameters ϵ. The corresponding entropy function is a function of the values u of the scalar fields, the size v of AdS_2 and S^2, the electric charge q conjugate to the variables ϵ, background charge parameter Q and cosmological constant μ. Then we fixed the u and v by extremizing the entropy function. Furthermore the entropy itself is given by the value of the entropy function at the horizon. As we know the entropy of black hole in two dimensions which is obtained by Cardy formula is proportional to central charge c and the central charge is proportional to the Q^2. One can see in (8) and (18) the black hole entropy is proportional to Q^2. Therefore our results about the black hole entropy cover the black hole entropy from several papers. In addition we obtained the correction to the entropy. Interesting problem here, is to use the entropy function formalism for higher dimensional black holes [34] or other two dimensional black holes such as [35].

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